Algebra 1 - Module 7 – Functions and Modeling

NATIONAL TRAINING NETWORK

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Big Idea
The properties of numbers can be used to interpret, analyze, build and graph multiple functions that represent mathematical and real-world situations. Working with various function families, students will begin to investigate the graphs of simple inverse functions, focusing on the inverses of linear functions and other cases, such as square root functions.
Vocabulary
Function, Range, Parent function, Linear Function , Quadratic Function, Exponential Function , Average Rate of Change, Cube root function, Square root function, End behavior, Recursive process, Piecewise Defined Function, Parameter, First Differences, Second Differences, Analytical model
Prior Learning
In Grades 6 – 8, students worked with defining, evaluating and comparing functions. They used functions to model relationships between quantities and also worked with exponential notation and applying the properties of integer exponents to solve equations. In prior Modules in Algebra I students have worked with families of functions such as linear, exponential, polynomial and quadratic functions.
Essential Questions
o How can functions be compared graphically to their parent functions o What scenarios might require models other than linear, exponential or quadratic? o What are the key features to other models, and what do they mean in the context of the problem? o How can the key features of a model be used to sketch a graph? o What are the general effects of transformations on models? o How can piecewise-defined functions be graphed? In what situations might these functions be applied? o What are the most common piecewise-defined functions, and when can they be used?
Competencies
o Students should be able to use graphical representations of other models, to include square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. o Students will be able to graph these functions by hand in simple cases and by using technology in more complex cases. o Students should be able to identify key features of their graphs. o Students should identify parent functions and describe or sketch the effects of simple transformations on those parent functions. • Informally identify the effects on different types of graphs when replacing with $f(x)$ with $f(x)+k$, $kf(x)$, $f(kx)$ and $f(x+k)$.
Misconceptions
o Students struggle to sketch graphs with an appropriate level of accuracy. o Students may find it difficult to determine which model should be used to represent a given situation. o The interpretation of key features in the context of a problem may be difficult for some students. o Piecewise functions, particularly step functions, are difficult to interpret and graph.
Resources from The Key Elements to Algebra Success - KEAS for Building the Conceptual Understanding of this Module

Strategies
Provide examples of real-world problems that can be modeled by writing an equation or inequality. Begin with simple equations and inequalities and build up to more complex equations in two or more variables that may involve quadratic, exponential or rational functions. Translating among different forms of expressions, equations and graphs helps students to understand some key connections among arithmetic, algebra and geometry. The reverse thinking technique (a process that allows working backwards from the answer to the starting point) can be very effective. Have students derive information about a function's equation, represented in standard, factored or general form, by investigating its graph. Completing the square is usually introduced for several reasons" to find the vertex of a parabola whose equation has been expanded; to look at the parabola through the lenses of translations of a "parent" parabola $y = x^2$; and to derive a quadratic formula. Completing the square is a very useful tool that will be used repeatedly by students in many areas of mathematics. Teachers should carefully balance traditional paper-pencil skills of manipulating quadratic expressions and solving quadratic equations along with an analysis of the relationship between parameters of quadratic equations and properties of their graphs. Start by inspecting equations such as $x^2 = 9$ that has two solutions, 3 and -3. Next, progress to equations such as $(x-7)^2 = 9$ by substituting $x-7$ for *x* and solving them either by "inspection" or by taking the square root on each side: $x-7=3$ and $x-7={}^{-}3$ $x=10$ and $x=4$ Graph both pairs of solutions (${}^{-}3$ and 3, 4 and 10) on the number line and notice that 4 and 10 are 7 units to the right of ${}^{-}3$ and 3. So, the substitution of $x - 7$ for x moved the solutions 7 units to the right. Next, graph the function $y=(x-7)^2-9$, pointing out that the *x*-intercepts are 4 and 10, and emphasizing that the graph is the translation of 7 units to the right and 9 units down from the position of the graph of the parent function $y = x^2$ that passes through the origin (0, 0). Generate more equations of the form $y = a(x-h)^2 + k$ and compare their graphs using a graphing technology. Highlight and compare different approaches to solving the same problem. Use technology to recognize that two different expressions or equations may represent the same relationship. For example, since $x^2 - 10x + 25 = 0$ can be rewritten as $(x-5)(x-5)=0$ or $(x-5)^2 = 0$, these are all representations of the same equation that has a double solution $x = 5$. Support it by putting all expressions into graphing calculator. Compare their graphs and generate their tables displaying the same output values for each expression. Guide students in transforming a quadratic equation in standard form, $0 = ax^2 + bx + c$, to the vertex form $0 = a(x-h)^2 + k$ by separating your examples into groups

with $a=1$ and $a\neq 1$ and have students guess the number that needs to be added to the binomials of the type x^2+6x, x^2-2x, x^2+9x, $x^2-\frac{2}{3}x$ to form complete square of the binomial $(x-m)^2$. Then generalize the process by showing the expression $(b/2)^2$ that has to be added to the binomial x^2+bx. Completing the square for an expression whose x^2 coefficient is not 1 can be complicated for some students. Present multiple examples of the type $0=2x^2-5x-9$ to emphasize the logic behind every step, keeping in mind that the same process will be used to complete the square in the proof of the quadratic formula.

Discourage students from giving a preference to a particular method of solving quadratic equations. Students need experience in analyzing a given problem to choose an appropriate solution method before their computations become burdensome. Point out that the Quadratic Formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is a universal tool that can solve any quadratic equation; however, it is not reasonable to use the Quadratic Formula when the quadratic equation is missing either a middle term, bx, or a constant term, c. When it is missing a constant term, (e.g., $3x^2-10x=0$) a factoring method becomes more efficient. If a middle term is missing (e.g., $2x^2-15=0$), a square root method is the most appropriate. Stress the benefit of memorizing the Quadratic Formula and flexibility with a factoring strategy. Introduce the concept of discriminants and their relationship to the number and nature of the roots of quadratic equations. Offer students examples of a quadratic equation, such as $x^2+9=0$. Since the graph of the quadratic function $y=x^2+9$ is situated above the x–axis and opens up, the graph does not have x–intercepts and therefore, the quadratic equation does not have real solutions.

This unit also focuses on linking expressions and functions, i.e., creating connections between multiple representations of functional relations – the dependence between a quadratic expression and a graph of the quadratic function it defines, and the dependence between different symbolic representations of exponential functions. Teachers need to foster the idea that changing the forms of expressions, such as factoring or completing the square, or transforming expressions from one exponential form to another, are not independent algorithms that are learned for the sake of symbol manipulations. They are processes that are guided by goals (e.g., investigating properties of families of functions and solving contextual problems).

Factoring methods that are typically introduced in elementary algebra and the method of completing the square reveals attributes of the graphs of quadratic functions, represented by quadratic equations.

- The solutions of quadratic equations solved by factoring are the x – intercepts of the parabola or zeros of quadratic functions.
- A pair of coordinates (h, k) from the general form $f(x) = a(x – h)^2 + k$ represents the vertex of the parabola, where h represents a horizontal shift and k represents a vertical shift of the parabola $y = x^2$ from its original position at the origin.
- A vertex (h, k) is the minimum point of the graph of the quadratic function if a › 0 and is the maximum point of the graph of the quadratic function if a ‹ 0. Understanding an algorithm of completing the square provides a solid foundation for deriving a quadratic formula.

Explore various families of functions and help students to make connections in terms of general features. For example, just as the function $y = (x + 3)^2 – 5$ represents a translation of the function $y = x$ by 3 units to the left and 5 units down, the same is true for the function $y = | x + 3 | − 5$ as a translation of the absolute value function $y = | x |$.

Discover that the factored form of a quadratic or polynomial equation can be used to determine the zeros, which in later courses can be used to identify maxima, minima and end behaviors.

Use various representations of the same function to emphasize different characteristics of that function. For example, the y-intercept of the function $y = x^2 - 4x - 12$ is easy to recognize as (0, -12). However, rewriting the function as $y = (x-6)(x+2)$ reveals zeros at (6, 0) and at ($^{-}$2, 0). Furthermore, completing the square allows the equation to be written as $y = (x-2)^2 - 16$, which shows that the vertex (and minimum point) of the parabola is at (2, $^{-}$16).

Use graphing calculators or computers to explore the effects of a constant in the graph of a function. For example, students should be able to distinguish between the graphs of $y = x^2$, $y = 2x^2$, $y = x^2 + 2$, $y = (2x)^2$, and $y = (x+2)^2$. This can be accomplished by allowing students to work with a single parent function and examine numerous parameter changes to make generalizations.

Additional Manipulatives/Tools/Resources to Enhance the Module	

Performance Tasks	
Table Tiling http://map.mathshell.org.uk/materials/tasks.php?taskid=283&subpage=expert	Sorting Functions http://map.mathshell.org.uk/materials/tasks.php?taskid=264&subpage=apprentice
Two Squares are Equal http://www.illustrativemathematics.org/illustrations/618	Building a General Quadratic Function http://www.illustrativemathematics.org/illustrations/505
Medieval Archer http://www.illustrativemathematics.org/illustrations/695	Building an explicit quadratic function by composition http://www.illustrativemathematics.org/illustrations/744

CCSS-Mathematics Content Standards	Example
A.APR.B.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial. i) Tasks are limited to quadratic and cubic polynomials in which linear and quadratic factors are available. For example, find the zeros of $(x-2)(x^2-9)$.	Find the zeros of a polynomial when the polynomial is factored. Then use the zeros to sketch the graph. Example 1: Given the function $y = 2x^2 + 6x - 3$, list the zeros of the function and sketch the graph. Example 2: Sketch the graph of the function $f(x) = (x+5)^2$. What is the multiplicity of the zeros of this function? How does the multiplicity relate to the graph of the function? Example 3: For a certain polynomial function, $x = 3$ is a zero with multiplicity two, $x = 1$ is a zero with multiplicity three, and $x = -3$ is a zero with multiplicity one. Write a possible equation for this function and sketch its graph.
F.IF.B.4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. ★	Students should be able to move fluidly between graphs, tables, words, and symbols and understand the connections between the different representations. For example, when given a table and graph of a function that models a real-life situation, explain how the table relates to the graph and vise versa. Also explain the meaning of the characteristics of the graph and table in the context of the problem as follows: • Quadratics – x-intercepts/zeroes, y-intercepts, vertex, intervals of increase/decrease, the effects of the coefficient of x^2 on the concavity of the graph, symmetry of a parabola.

	Example: Below is a table that represents the relationship between daily income, *I*, for an amusement park and the number of paying visitors in thousands, *n*. *n* | *I* 0 | 0 1 | 5 2 | 8 3 | 9 4 | 8 5 | 5 6 | 0 a) What are the *x*-intercepts and *y*-intercepts and explain them in the context of the problem? b) Identify any maximums or minimums and explain their meaning in the context of the problem. c) What pattern of change do these ordered pairs develop? Explain. d) Is the pattern and/or graph of the data symmetrical? How do you know? e) Describe the intervals of increase and decrease and explain them in the context of the problem. Include piecewise, square root, cube root, absolute value, and step functions.
F.IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. ★	From a graph students will identify the domain. In context, students will identify the domain, stating any restrictions and why they are restrictions. At this level, focus on linear and exponential functions. Example: If Jennifer buys a cell phone and the plan she decided upon charged her $50 for the phone and $0.10 for each minute she is on the phone. What would be the appropriate domain that describes this relationship? Describe what is meant by the point (10, 51).
F.IF.B.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. (★)	In addition to find average rates of change from functions given symbolically, graphically, or in a table, students may collect data from experiments or simulations and find the average rates of change for the function modeling the situation.

<table>
<tr>
<td>F.IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
b) Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.</td>
<td>This standard should be seen as related to F.IF.B.4 with the key difference being students can create graphs, by hand and using technology, from the symbolic function in this standard.

In Algebra I for F.IF.7b, compare and contrast absolute value, step and piecewise- defined functions with linear and quadratic functions. Include comparisons of two functions presented algebraically. Highlight issues of domain, range, and usefulness when examining piecewise- defined functions.

Key characteristics include but are not limited to maxima, minima, intercepts, symmetry, end behavior, and asymptotes. Students may use graphing calculators or programs, spreadsheets, or computer algebra systems to graph functions.

Example 1: The all-star kicker kicks a field goal for the team and the path of the ball is modeled by $f(t) = 4.9t^2 + 20t$. Find the realistic maximum and minimum values for the path of the ball and describe what each means in the context of this problem.

Example 2: Describe key characteristics of the graph of $f(x) = \left| x - 3 \right| + 5$.</td>
</tr>
<tr>
<td>F.BF.A.1 Write a function that describes a relationship between two quantities.★
a) Determine an explicit expression, a recursive process, or steps for calculation from a context.</td>
<td>Recognize when a relationship exists between two quantities and write a function to describe them. Use steps, the recursive process, to make the calculations from context in order to write the explicit expression that represents the relationship.</td>
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<td>F.BF.B.3. Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. ~~Include recognizing even and odd functions from their graphs and algebraic expressions for them.~~</td>
<td>Identify the effect transformations have on functions in multiple modalities or ways. Student should be fluent with representations of functions as equations, tables, graphs, and descriptions. They should also understand how each representation of a function is related to the others. For example, the equation of the line is related to its graph, table, and when in context, the problem being solved.

Ex. Fill in all missing components to the below table.
<table>
<tr><th>Description of Change</th><th>Original Function Output</th><th></th><th>Multiply the original function output by 5</th></tr>
<tr><td>x</td><td>$f(x)$</td><td>$f(x)+5$</td><td></td></tr>
<tr><td>-3</td><td>9</td><td></td><td></td></tr>
<tr><td>-2</td><td>4</td><td></td><td></td></tr>
<tr><td>-1</td><td>1</td><td></td><td>5</td></tr>
<tr><td>0</td><td>0</td><td></td><td></td></tr>
<tr><td>1</td><td>1</td><td></td><td></td></tr>
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<tr>
<td></td>
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<table>
<tr><td>2</td><td>4</td><td>9</td><td></td></tr>
<tr><td>3</td><td>9</td><td></td><td></td></tr>
</table>
a) Graph and label each of the function outputs with the corresponding x-values on the same set of axis in three different colors.
b) Explain the relationship that exists between the original function and the transformed functions.
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<td>F.LE.A.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a ~~polynomial function.~~</td>
<td>
When students compare graphs of various functions, such as linear, exponential, quadratic, and polynomial they should see that any values that increase exponentially eventually increases or grows at a faster rate than values that increase linearly, quadratically, or any polynomial function. At this level, limit to linear, exponential, and quadratic functions; ~~general polynomial functions~~ are not addressed.

Example: Carrie and Elizabeth applied for a job at the local seafood market. Carrie asked for $2 and hour, but the boss proposed giving them $.010 for the first hour, $.020 for the second hour, $.040 for the third hour, $.080, and so on.

Below is a table of the hours worked per week and the pay for each hour for each of the two pay plans. When would you want to use Carrie's plan and when would you use the Boss's plan? Why?
<table>
<tr><td>Hours worked in a week</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>Earnings for Carrie's plan</td><td>2</td><td>4</td><td>6</td><td>8</td><td>10</td><td>12</td><td>14</td><td>16</td><td>18</td><td>20</td></tr>
<tr><td>Earnings for Boss's plan</td><td>0.10</td><td>0.30</td><td>0.70</td><td>1.50</td><td>3.10</td><td>6.30</td><td>12.70</td><td>25.50</td><td>51.10</td><td>102.30</td></tr>
</table>
</td>
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<tr>
<td>A.REI.D.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive</td>
<td>Construct an argument to demonstrate understanding that the solution to every equation can be found by treating each side of the equation as separate functions that are set equal to each other, $f(x) = g(x)$. Allow $y_1 = f(x)$ and $y_2 = g(x)$ and find their intersection(s). The x-coordinate of the point of intersection is the value at which these two functions are equivalent, therefore the solution(s) to the original equation. Students should understand that this can be treated as a system of equations and should also include the use of technology to justify their argument using graphs, tables of values, or successive approximations.</td>
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</table>

approximations. Include cases where *f*(*x*) and/or *g*(*x*) are linear, polynomial, rational, absolute value, ~~exponential, and logarithmic functions.~~★	Ex. John and Jerry both have jobs working at the town carnival. They have different employers, so their daily wages are calculated differently. John's earnings are represented by the equation, $p(x) = 2x$ and Jerry's by $g(x) = 10 + 0.25x$. a. What does the variable x represent? b. If they begin work next Monday, Michelle told them that Friday would be the only day they made the same amount of money. Is she correct in her assumption? Explain your reasoning. c. When will Jerry earn more money than John? When will John earn more money than Jerry? During what day will their earnings be the same? Justify your conclusions.

NYS Performance Level Descriptors					
Domain	**NYS Level 5**	**NYS Level 4**	**NYS Level 3**	**NYS Level 2**	**NYS Level 1**
Arithmetic with Polynomials and Rational Expressions (A-APR)	**Explain and/or show generally** that polynomials are closed under addition, subtraction, and multiplication. **Determine and use the zeros of any polynomial function** to sketch its graph, generate graphs and expressions for multiple functions, given particular zeros, and explain the significance of the zeros.	Perform addition, subtraction, and multiplication with polynomials **and demonstrate that polynomials are closed under the three operations.** **Identify zeros of quadratic and cubic polynomials** and use the zeros to graph the function. **Explain the relationship** between a function and its zeros.	Perform **addition, subtraction, and multiplication on polynomials.** **Identify zeros of quadratic polynomials** and use the zeros to graph the function.	Perform **addition and subtraction** with linear expressions. Given a **linear polynomial**, construct a graph of the function and identify its zero.	Perform **addition** with linear expressions.
Interpreting Functions (F-IF continued)	Accurately sketch graphs, showing key features, given a verbal description of the relationship, **including piece-wise defined and step functions.** Estimate, calculate, and interpret the average rate of change **in terms of a context** over a specified interval, including linear, quadratic, square root, cube root, piece-wise defined, and exponential functions with domains in the real numbers.	**Accurately sketch and create graphs using technology and interpret** key features of graphs and tables given a verbal description of the relationship, **including square root and cube root functions with domains in real numbers.** Estimate, calculate, and interpret the average rate of change over a specified interval, **including linear, quadratic, square root, cube root, piece-wise defined and exponential functions with domains in the integers.** Use the process of factoring and completing the square in a quadratic function **to show zeros, extreme values, and symmetry of the graph and interpret these in terms of a context.**	**Accurately sketch and create graphs using technology and identify** key features of graphs, given a verbal description of the relationship**, including linear, quadratic, and exponential functions with domains in the integers.** Calculate the average rate of change over a specified interval from a graph, **including linear, quadratic, and exponential functions with domains in the integers.** Use the process of factoring **to show zeros and symmetry of a graph.**	**Graph linear and quadratic functions** and identify key features visible within the "standard zoom" (-10 to 10 calculator window) by hand or technology. Calculate the rate of change **of a linear function from a graph or table.** Graph quadratic functions using **technology and identify their roots.**	**Identify the properties of linear functions** represented algebraically, graphically, or numerically in tables. **Identify the rate of change** given the symbolic representation of a linear function. Distinguish between graphs of **increasing and decreasing** linear functions. Identify *x*-intercepts of a quadratic function, **given its graph.**

(F-IF continued)	Compare properties of two functions with each represented in a different way (i.e., algebraically, graphically, numerically in tables, or by verbal descriptions), including linear, quadratic, square root, cube root, piecewise-quadratic, and exponential functions **with domains in the real numbers.**	Compare properties of two functions with each represented in a different way (i.e., algebraically, graphically, numerically in tables, or by verbal descriptions), **including linear, quadratic, square root, cube root, piecewise-quadratic, and exponential functions with domains in the integers.**	Compare **properties** of two functions with each **represented in a different way** (i.e., algebraically, graphically, or numerically in tables), **including linear, quadratic, and exponential functions with domains in the integers.**	Compare **qualitative descriptions** of two **linear functions represented in the same way** (i.e., algebraically, graphically, or numerically in tables).	
Building Functions (F-BF)	Given the equation of a transformed linear or quadratic function, **create an appropriate graph and interpret the transformations.**	Identify the effect on a graph of **replacing $f(x)$ with $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$. Find the value of k given the graphs.**	Identify the effect on a graph of **replacing $f(x)$ with $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative integers).**	Identify the effect on a graph of **replacing $f(x)$ with $f(x) + k$ where k is a positive or negative integer and replacing $f(x)$ with $kf(x)$ where k is a positive integer.**	Identify the effect on a graph of **replacing $f(x)$ with $f(x) + k$ where k is a positive integer.**
Linear, Quadratic, and Exponential Models (F-LE)	Interpret **changes in parameters** based on the comparison of two functions in terms of a real-world context.	Identify situations in which a quantity grows or decays at **a constant percent rate per unit interval relative to another.** **Interpret the parameters** (i.e., slope or growth factor) in a linear, quadratic, or exponential function in terms of a real-world context.	Identify situations in which one quantity changes at **a constant rate per unit interval relative to another.** Identify and **distinguish between situations that can be modeled with linear functions and exponential functions.** **Identify the slope and y-intercept in a linear function** in terms of a real-world context.	**Using a graph**, show that a quantity increasing exponentially grows faster than a quantity increasing linearly.	

Works Referenced in the Development of the Module	
Common Core State Standards Initiative http://www.corestandards.org/	North Carolina Mathematics Wiki http://maccss.ncdpi.wikispaces.net/
Illustrative Mathematics Project http://illustrativemathematics.org	PARCC http://parcconline.org/
Mathematics Assessment Project http://map.mathshell.org.uk/materials/index.php	Smarter Balanced Assessment Consortium http://www.smarterbalanced.org/
Ohio Department of Education http://education.ohio.gov/GD/Templates/Pages/ODE/ODEPrimary.aspx?page=2&TopicRelationID=1704	Utah Education Network http://www.uen.org/commoncore/